Mechanika@FJFI - praktický úvod do modelování fyzikálních dějů

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Table of Contents

Table of Contents

Motivation

Scientific problem

Theory, **Numerical simulation**, Experiment

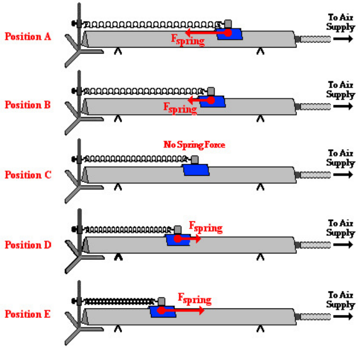


Figure: Motion of a Mass on a Spring

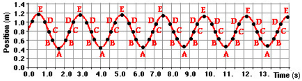


Figure: SHO position analysis @ [Hen20]

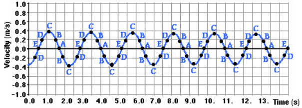


Figure: SHO velocity analysis @ [Hen20]

I. Štoll: Mechanics & ElMa

Force field to be inserted into the Newton's law of motion $\mathbf{F} = m \cdot \mathbf{a}$

$\mathbf{F} =$

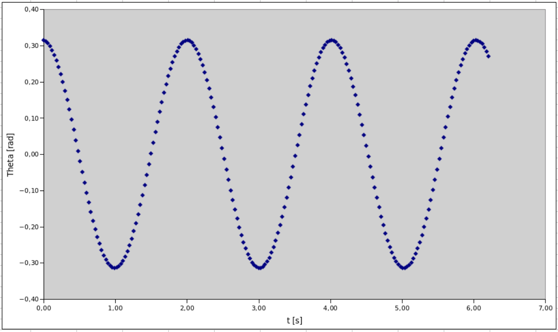
- $=0$ (zero force)
- $=$konst (constant force)
- $= m \cdot \mathrm{g}$ (gravitational field)
- $= -\mathrm{k} \cdot x$ (simple harmonic oscillator)
- $= -\mathrm{b} \cdot v$ (friction force)
- generally $f(\mathbf{r}, \mathbf{v}, \mathrm{t})$
- $= q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ (Lorentz force)

and so on ..

and so on ...

Sreenshot: Pendulum basic @ spreadsheet

	A	B	C	D	E	I	J	K	L	M	N
1	t	F	eps	omega	theta	m	1,00	kg	T	2,01	s
2	[s]	[N]	rad/s*2	rad/s	rad	g	9,81	m/s**2			
3	0,00			0,00	0,31	dt	0,03	s			
4	0,03	−3,03	−3,03	−0,09	0,31	l	1,00	m			
5	0,06	−3,01	−3,01	−0,18	0,31						
6	0,09	−2,96	−2,96	−0,27	0,30						
7	0,12	−2,88	−2,88	−0,36	0,29						
8	0,15	−2,78	−2,78	−0,44	0,27						
9	0,18	−2,66	−2,66	−0,52	0,26						
10	0,21	−2,51	−2,51	−0,59	0,24						
11	0,24	−2,34	−2,34	−0,66	0,22						
12	0,27	−2,15	−2,15	−0,73	0,20						
13	0,30	−1,94	−1,94	−0,79	0,18						
14	0,33	−1,71	−1,71	−0,84	0,15						
15	0,36	−1,47	−1,47	−0,88	0,12						
16	0,39	−1,21	−1,21	−0,92	0,10						
17	0,42	−0,94	−0,94	−0,95	0,07						
18	0,45	−0,66	−0,66	−0,97	0,04						
19	0,48	−0,38	−0,38	−0,98	0,01						
20	0,51	−0,09	−0,09	−0,98	−0,02						
21	0,54	0,20	0,20	−0,98	−0,05						
22	0,57	0,49	0,49	−0,96	−0,08						
23	0,60	0,77	0,77	−0,94	−0,11						
24	0,63	1,04	1,04	−0,91	−0,13						
25	0,66	1,31	1,31	−0,87	−0,16						
26	0,69	1,56	1,56	−0,82	−0,18						
27	0,72	1,80	1,80	−0,77	−0,21						
28	0,75	2,02	2,02	−0,71	−0,23						
29	0,78	2,22	2,22	−0,64	−0,25						
30	0,81	2,40	2,40	−0,57	−0,26						
31	0,84	2,57	2,57	−0,49	−0,28						
32	0,87	2,71	2,71	−0,41	−0,29						
33	0,90	2,82	2,82	−0,32	−0,30						
34	0,93	2,91	2,91	−0,24	−0,31						
35	0,96	2,98	2,98	−0,15	−0,31						
36	0,99	3,02	3,02	−0,06	−0,31						
37	1,02	3,04	3,04	0,03	−0,31						
38	1,05	3,03	3,03	0,13	−0,31						
39	1,08	2,99	2,99	0,21	−0,30						



▸ See example

Sreenshot: Pendulum basic @ processing

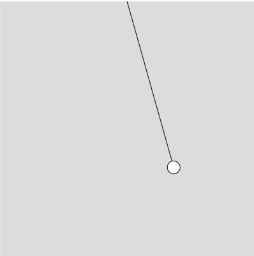
https://editor.p5js.org/vojtech.svob/sketches/vTEaAkgs

Pendulum - basic version by vojtech.svob

sketch.js — Saved: 3 minutes ago

```
function setup() {
  createCanvas(400, 400);
  m=2
  l=2.705
  g=9.814
  dt=0.02
  t=0
  theta = 3.14/10; // Pendulum initial angle theta
  omega = 0; // Initial angular velocity
  C = 2; // Center point
}

function draw() {
  background(220);
  // physics
  t = t + dt;
  F=-m*g*sin(theta)
  epsilon = (F/m)/l; //angular acceleration
  omega = omega + epsilon * dt;
  theta = theta + omega * dt;
  xp = C - l * sin(theta); // X coordinate of pendulum ball
  yp = l * cos(theta); // Y coordinate of pendulum ball
  //draw it
  ppm=100 //scale it to the canvas (from meters to pixels)
  line(C*ppm, 0, xp*ppm, yp*ppm);
  ellipse(xp*ppm, yp*ppm, 20, 20);
}
```

Preview



See example

Objectives

(World) Pendulum ... as a gate to physics

Numerical simulations point of view

- A comprehensive, as simple as possible numerical approach to the Pendulum problem using Euler scheme for solving ordinary differential equations (ODE) developed under various Computer Algebraic Systems:
  - spreadsheet (Excel, LibreOffice Calc, Google, gnumeric),
  - p5* processing,
  - jupyter notebook (python),
  - octave (matlab).
- Wide range of simple examples (ready to be used for education)
- Way to avoid the complex math problems (ODE) in the (early) physics education.

Outline of the talk

Table of Contents

Initial value problem

Let's have a general force field $F(t, x, v)$ applying on an object of a mass m, having some initial conditions t_0, v_0, x_0:

- Differential solution: having dt time progress: $a = F/m$, then $v(t) = \int_{t_0}^{t} a dt$, and $x(t) = \int_{t_0}^{t} v dt$
- Discrete solution: having Δt time progress, in principal, we are looking for a time series of object position $(t_0, x_0), (t_1, x_1), ..(t_n, x_n)$: $a_i = F_i/m$, then $v_{i+1} = v_i + a \cdot \Delta t$, and $x_{i+1} = x_i + v_i \cdot \Delta t$

Discrete solution - towards algorithmization

Recurring principle/algorithm

ideal for computer algebraic systems

Having Δt time progress, in principal, we are looking for a time series of object position $(t_0, x_0), (t_1, x_1), ..(t_n, x_n)$: $a_i = F_i/m$, then $v_{i+1} = v_i + a \cdot \Delta t$, and $x_{i+1} = x_i + v_i \cdot \Delta t$

time	$F(t, x, v)$	$a(t)$	$v(t)$ calculation	$x(t)$ calculation
t_0	$F_0 = F(t_0, x_0, v_0)$	$a_0 = F_0/m$	v_0 (initial cond.)	x_0 (initial cond.)
$t_1 = t_0 + \Delta t$	$F_1 = F(t_1, x_1, v_1)$	$a_1 = F_1/m$	$v_1 = v_0 + a_1 \Delta t$	$x_1 = x_0 + v_1 \Delta t$
$t_2 = t_1 + \Delta t$	$F_2 = F(t_2, x_2, v_2)$	$a_2 = F_2/m$	$v_2 = v_1 + a_2 \Delta t$	$x_2 = x_1 + v_2 \Delta t$
..	..	..	..	..
$t_n = t_{n-1} + \Delta t$	$F_n = F(t_n, x_n, v_n)$	$a_n = F_n/m$	$v_n = v_{n-1} + a_n \Delta t$	$x_n = x_{n-1} + v_n \Delta t$

Euler method solving ODE - the principle

Let an initial value problem be specified:

$$\dot{y} = f(t, y), \quad y(t_0) = y_0$$

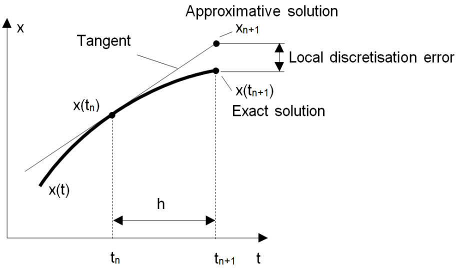


Figure: credit:[Sza14]

$$y_{n+1} = y_n + h\ f(t_n, y_n),$$
$$t_{n+1} = t_n + h$$

Euler method solving ODE - repetition (loop)

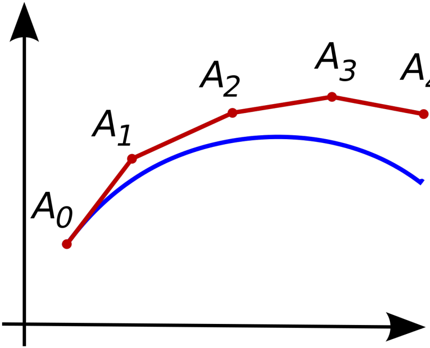


Figure: credit:[Wik20a]

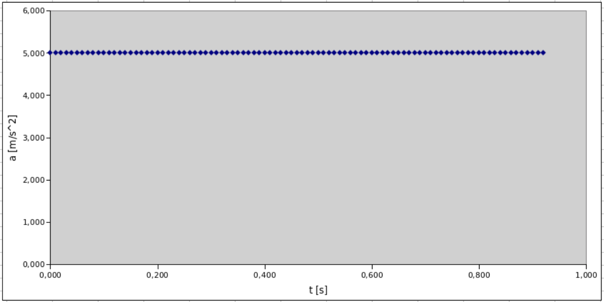
Sreenshot: Let's dive into a problem

0^{th} order ODE: Constant force

$F_{ext} = k$

	A	B	C
1	t	F	a
2	0,000	10,000	5,000
3	0,010	10,000	5,000
4	0,020	10,000	5,000
5	0,030	10,000	5,000
6	0,040	10,000	5,000
7	0,050	10,000	5,000
8	0,060	10,000	5,000
9	0,070	10,000	5,000
10	0,080	10,000	5,000
11	0,090	10,000	5,000
12	0,100	10,000	5,000
13	0,110	10,000	5,000
14	0,120	10,000	5,000
15	0,130	10,000	5,000
16	0,140	10,000	5,000
17	0,150	10,000	5,000
18	0,160	10,000	5,000
19	0,170	10,000	5,000
20	0,180	10,000	5,000
21	0,190	10,000	5,000
22	0,200	10,000	5,000
23	0,210	10,000	5,000
24	0,220	10,000	5,000
25	0,230	10,000	5,000
26	0,240	10,000	5,000
27	0,250	10,000	5,000
28	0,260	10,000	5,000
29	0,270	10,000	5,000
30	0,280	10,000	5,000
31	0,290	10,000	5,000
32	0,300	10,000	5,000
33	0,310	10,000	5,000
34	0,320	10,000	5,000
35	0,330	10,000	5,000
36	0,340	10,000	5,000
37	0,350	10,000	5,000

	I	J
1	dt	0,010
2	m	2,000
3	F_ext	10,000



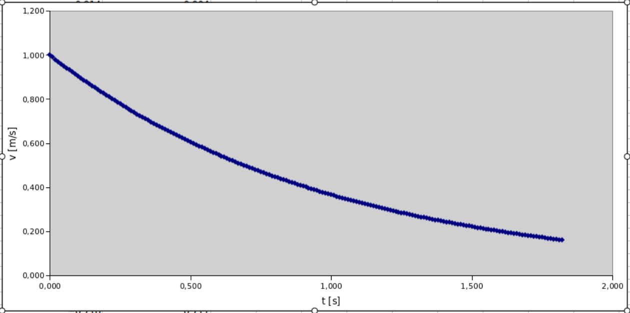
See example

Sreenshot: Let's dive into a problem

1^{st} order ODE: Friction force

$$F_{ext} = -b \cdot v$$

	A	B	C	D	I	J
1	t	F	a	v	b	1,000
2	0,000			1,000	dt	0,010
3	0,010	−1,000	−1,000	0,990	m	1,000
4	0,020	−0,990	−0,990	0,980		
5	0,030	−0,980	−0,980	0,970		
6	0,040	−0,970	−0,970	0,961		
7	0,050	−0,961	−0,961	0,951		
8	0,060	−0,951	−0,951	0,941		
9	0,070	−0,941	−0,941	0,932		
10	0,080	−0,932	−0,932	0,923		
11	0,090	−0,923	−0,923	0,914		
12	0,100	−0,914				
13	0,110	−0,904				
14	0,120	−0,895				
15	0,130	−0,886				
16	0,140	−0,878				
17	0,150	−0,869				
18	0,160	−0,860				
19	0,170	−0,851				
20	0,180	−0,843				
21	0,190	−0,835				
22	0,200	−0,826				
23	0,210	−0,818				
24	0,220	−0,810				
25	0,230	−0,802				
26	0,240	−0,794				
27	0,250	−0,786				
28	0,260	−0,778				
29	0,270	−0,770				
30	0,280	−0,762				
31	0,290	−0,755				
32	0,300	−0,747				
33	0,310	−0,740				
34	0,320	−0,732				
35	0,330	−0,725				
36	0,340	−0,718				
37	0,350	−0,711	−0,711	0,703		
38	0,360	−0,703	−0,703	0,696		



See example

Table of Contents

Free fall set-up

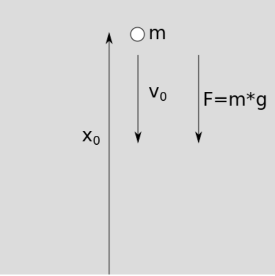


Figure: Experiment set-up

Equation of motion:

$$
\begin{aligned}
F_{ext} &= -mg, \\
a &= F_{ext}/m \\
dv/dt &= a \\
dx/dt &= v
\end{aligned}
$$

Table of Contents

A spreadsheet approach

time	$F(t,x,v)$	$a(t)$	$v(t)$ calculation	$x(t)$ calculation
t_0	$F_0 = F(t_0, x_0, v_0)$	$a_0 = F_0/m$	v_0 (initial cond.)	x_0 (initial cond.)
$t_1 = t_0 + \Delta t$	$F_1 = F(t_1, x_1, v_1)$	$a_1 = F_1/m$	$v_1 = v_0 + a_1 \Delta t$	$x_1 = x_0 + v_1 \Delta t$
$t_2 = t_1 + \Delta t$	$F_2 = F(t_2, x_2, v_2)$	$a_2 = F_2/m$	$v_2 = v_1 + a_2 \Delta t$	$x_2 = x_1 + v_2 \Delta t$
..	..	..	..	..
$t_n = t_{n-1} + \Delta t$	$F_n = F(t_n, x_n, v_n)$	$a_n = F_n/m$	$v_n = v_{n-1} + a_n \Delta t$	$x_n = x_{n-1} + v_n \Delta t$

Let us have a force in a cell L2, object mass in a cell I2, time advance in a cell I4, initial height in a cell E4 and initial velocity in a cell D4, then

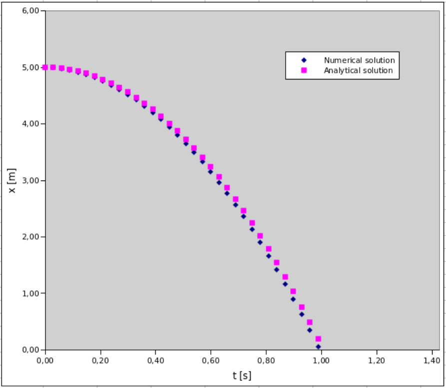
row	column A	column B	column C	column D	column E
4	0	-L2	B4/I2	any number (v_0 initial cond.)	any number (x_0 initial cond.)
5	A4+I4	-L2	B5/I2	D4+C5*I4	E4+D5*I4
6	A5+I4	-L2	B6/I2	D5+C6*I4	E5+D6*I4
7..N-1	..	..	..	..	..
N	A(N-1)+I4	-L2	BN/I2	D(N-1)+CN*I4	E(N-1)+DN*I4

So it is possible to specify only row #5 and then use copy row #5 and paste special to the consequent rows from #6 to #N.

▸ See example

Sreenshot: Free fall (numerical and analytical comparison)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Discrete solution					Analytical solution										
2	t	F	a	v	x			m	1,00	kg	F	9,81	N			
3	[s]	[N]	[m/s/s]	[m/s]	m			g	9,81	m/s/s	t_fall	1,01	s			
4	0,00			0,00	5,00	5,00		dt	0,03	s						
5	0,03	−9,81	−9,81	−0,29	4,99	5,00										
6	0,06	−9,81	−9,81	−0,59	4,97	4,98										
7	0,09	−9,81	−9,81	−0,88	4,95	4,96										
8	0,12	−9,81	−9,81	−1,18	4,91	4,93										
9	0,15	−9,81	−9,81	−1,47	4,87	4,89										
10	0,18	−9,81	−9,81	−1,77	4,81	4,84										
11	0,21	−9,81	−9,81	−2,06	4,75	4,78										
12	0,24	−9,81	−9,81	−2,35	4,68	4,72										
13	0,27	−9,81	−9,81	−2,65	4,60	4,64										
14	0,30	−9,81	−9,81	−2,94	4,51	4,56										
15	0,33	−9,81	−9,81	−3,24	4,42	4,47										
16	0,36	−9,81	−9,81	−3,53	4,31	4,36										
17	0,39	−9,81	−9,81	−3,83	4,20	4,25										
18	0,42	−9,81	−9,81	−4,12	4,07	4,13										
19	0,45	−9,81	−9,81	−4,41	3,94	4,01										
20	0,48	−9,81	−9,81	−4,71	3,80	3,87										
21	0,51	−9,81	−9,81	−5,00	3,65	3,72										
22	0,54	−9,81	−9,81	−5,30	3,49	3,57										
23	0,57	−9,81	−9,81	−5,59	3,32	3,41										
24	0,60	−9,81	−9,81	−5,89	3,15	3,23										
25	0,63	−9,81	−9,81	−6,18	2,96	3,05										
26	0,66	−9,81	−9,81	−6,47	2,77	2,86										
27	0,69	−9,81	−9,81	−6,77	2,56	2,66										
28	0,72	−9,81	−9,81	−7,06	2,35	2,46										
29	0,75	−9,81	−9,81	−7,36	2,13	2,24										
30	0,78	−9,81	−9,81	−7,65	1,90	2,02										
31	0,81	−9,81	−9,81	−7,95	1,66	1,78										
32	0,84	−9,81	−9,81	−8,24	1,42	1,54										
33	0,87	−9,81	−9,81	−8,53	1,16	1,29										
34	0,90	−9,81	−9,81	−8,83	0,89	1,03										
35	0,93	−9,81	−9,81	−9,12	0,62	0,76										
36	0,96	−9,81	−9,81	−9,42	0,34	0,48										
37	0,99	−9,81	−9,81	−9,71	0,05	0,19										
38	1,02	−9,81	−9,81	−10,01	−0,25	−0,10										
39	1,05	−9,81	−9,81	−10,30	−0,56	−0,41										



▸ See example

A spreadsheet approach cont.

row	column A	column B	column C	column D	column E
4	0	-L2	B4/I2	any number (v_0 initial cond.)	any number (x_0 initial cond.)
5	A4+I4	-L2	B5/I2	D4+C5*I4	E4+D5*I4
6	A5+I4	-L2	B6/I2	D5+C6*I4	E5+D6*I4
7..N-1	..	..	..	..	..
N	A(N-1)+I4	-L2	BN/I2	D(N-1)+CN*I4	E(N-1)+DN*I4

A more convenient way is to name basic parameters, e.g. Let us have a force in a cell L2 named F, object mass in a cell I2 named m, time advance in a cell I4 named dt, initial height in a cell E4 and initial velocity in a cell D4, then

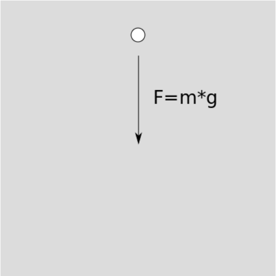
row	column A	column B	column C	column D	column E
4	0	$-F$	$B4/m$	any number (v_0 initial cond.)	any number (x_0 initial cond.)
5	$A4+dt$	$-F$	$B5/m$	$D4+C5*dt$	$E4+D5*dt$
6	$A5+dt$	$-F$	$B6/m$	$D5+C6*dt$	$E5+D6*dt$
7..N-1	..	..	..	..	..
N	$A(N-1)+dt$	$-F$	BN/m	$D(N-1)+CN*dt$	$E(N-1)+DN*dt$

See example

Table of Contents

A processing approach

```
function setup() {
  createCanvas(200, 500); //  width, height
  m=1 // [kg] mass of the object
  x=5 // initial position
  v=0 // initial velocity
  g=9.814 //[m/s^2] gravitational constant Lisbon
  F=-m*g
  dt=0.003 // [s] time advance
  t=0 // [s] initial time
}

function draw() {
  background(220); // try to comment it
  // Physics
  t=t+dt // time evolution
  a=F/m // acceleration "evolution"
  v=v+a*dt // velocity evolution
  x=x+v*dt // position evolution
  // Drawing
  // ... into canvas widthxheight and origin left-up corner
  x_canvas=height-x*100 // 1m=100pixels & rotate it upside-down
  circle(100,x_canvas,20)
  if ( x<=0 ) {F=0,x=0} //Good to stop it
}
```



See example

Sreenshot: Free fall

https://editor.p5js.org/vojtech.svob/sketches/p_VGqDX5

M C N Ggs TV@J Trelo Aktual KnowH GM Dg GW #0 GMrm Osobni Duše Galleries Viol YT Bck

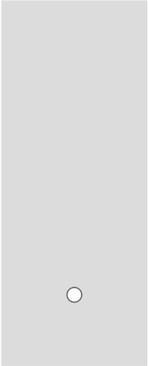
p5* File Edit Sketch Help

Auto-refresh Free fall by vojtech.svob

sketch.js

```
function setup() {
  createCanvas(200, 500); //  width, height
  m=1 // [kg] mass of the object
  x=5 // initial position
  v=0 // initial velocity
  g=9.814 //[m/s^2] gravitational constant Lisbon
  F=-m*g
  dt=0.003 // [s] time advance
  t=0 // [s] initial time
}

function draw() {
  background(220); // try to comment it
  // Physics
  t=t+dt // time evolution
  a=F/m // acceleration "evolution"
  v=v+a*dt // velocity evolution
  x=x+v*dt // position evolution
  // Drawing
  // ... into canvas widthxheight and origin left-up corner
  x_canvas=height-x*100 // 1m=100pixels & rotate it upside-down
  circle(100,x_canvas,20)
  if ( x<=1 ) {F=0,x=1} //Good to stop it
}
```

Preview

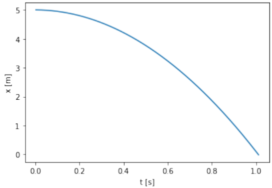


See example

Table of Contents

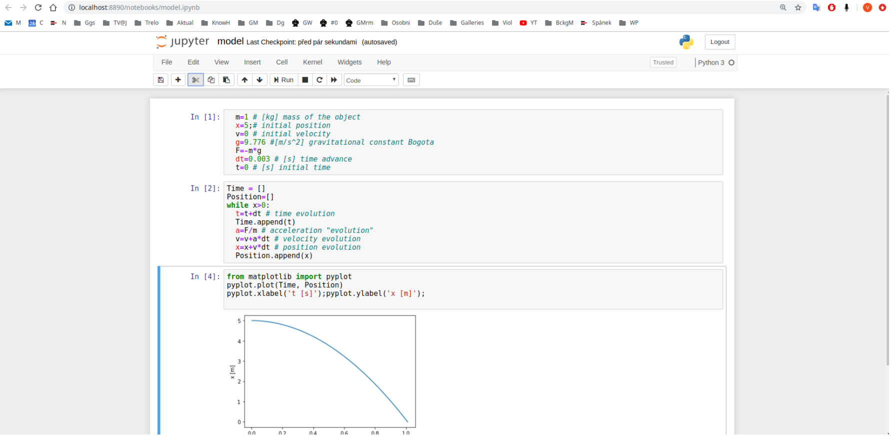
A python@Jupyter notebook approach

```
m=1 # [kg] mass of the object
x=5;# initial position
v=0 # initial velocity
g=9.814 #[m/s^2] gravitational constant Lisbon
F=-m*g
dt=0.003 # [s] time advance
t=0 # [s] initial time

Time = []
Position=[]
while x>0:
   t=t+dt # time evolution
   Time.append(t)
   a=F/m # acceleration "evolution"
   v=v+a*dt # velocity evolution
   x=x+v*dt # position evolution
   Position.append(x)

from matplotlib import pyplot
pyplot.plot(Time, Position)
pyplot.xlabel('t [s]'); pyplot.ylabel('x [m]');
```



See example

Sreenshot: Free fall



See example

Table of Contents

Simple harmonic motion

2.2 Pohyb kmitavý

2.2.1 Harmonický oscilátor

Uvažujme jednorozměrný pohyb částice hmotnosti m například podél osy x, na níž působí síla $F_x = -kx$, $k > 0$. Je to zřejmě síla konzervativní a částice má v tomto silovém poli potenciální energii

$$U(x) = \frac{1}{2}\,k\,x^2\,. \tag{1}$$

Aditivní konstantu jsme volili tak, aby potenciální energie byla nulová při $x = 0$. Částice má při pohybu určitou energii E, která se zachovává, je integrálem pohybu; přitom musí platit podmínka $E \geq U(x)$. Lze tedy říci, že se částice pohybuje v symetrické, parabolické potenciálové jámě a x leží v mezích $-A \leq x \leq A$, kde A je amplituda, největší výchylka (obr. 2.8).

Ukážeme, že taková částice bude vykonávat netlumené harmonické kmity s

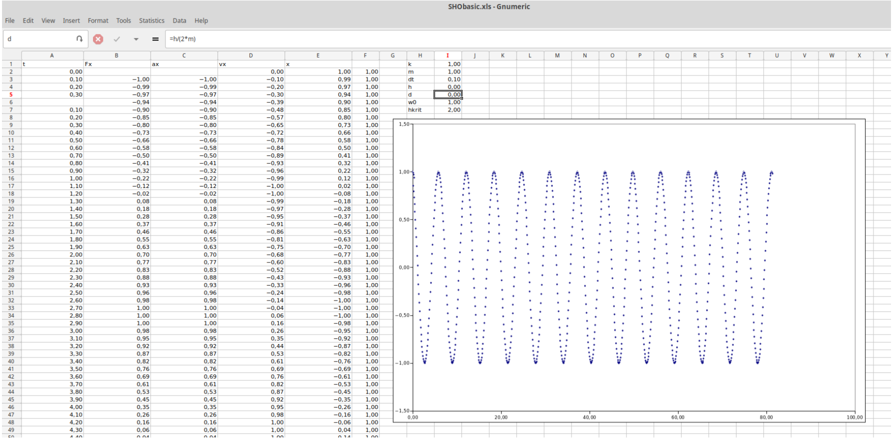
$$\omega_0 = \sqrt{\frac{k}{m}}\,. \tag{2}$$

Tato soustava se proto nazývá **harmonický oscilátor** a má zásadní význam ve všech oblastech fyziky. Její důležitost je v tom, že jakoukoli symetrickou potenciálovou jámu můžeme pro malé kmity vždy aproximovat jámou parabolickou a harmonický oscilátor tedy obecně popisuje libovolné kmitavé pohyby s malou amplitudou. Jeho důležitou vlastností je i to, že jeho vlastní úhlová frekvence (která je jediným parametrem charakterizujícím harmonický oscilátor) nezávisí na amplitudě a že perioda kmitů je vždy stejná nezávisle na počáteční výchylce.

Pohybovou rovnici harmonického oscilátoru

$$m\,\ddot{x} = -\,k\,x \tag{3}$$

Sreenshot: Basic SHO



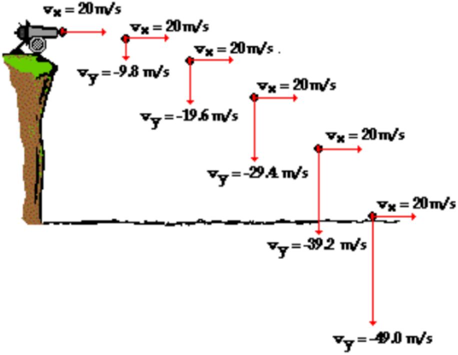
	A	B	C	D	E	F	G	H	I
1	t	Fx	ax	vx	x			k	1,00
2	0,00			0,00	1,00	1,00		m	1,00
3	0,10	−1,00	−1,00	−0,10	0,99	1,00		dt	0,10
4	0,20	−0,99	−0,99	−0,20	0,97	1,00		h	0,00
5	0,30	−0,97	−0,97	−0,30	0,94	1,00		d	0,00
6		−0,94	−0,94	−0,39	0,90	1,00		w0	1,00
7	0,10	−0,90	−0,90	−0,48	0,85	1,00		hkrit	2,00
8	0,20	−0,85	−0,85	−0,57	0,80	1,00			
9	0,30	−0,80	−0,80	−0,65	0,73	1,00			
10	0,40	−0,73	−0,73	−0,72	0,66	1,00			
11	0,50	−0,66	−0,66	−0,78	0,58	1,00			
12	0,60	−0,58	−0,58	−0,84	0,50	1,00			
13	0,70	−0,50	−0,50	−0,89	0,41	1,00			
14	0,80	−0,41	−0,41	−0,93	0,32	1,00			
15	0,90	−0,32	−0,32	−0,96	0,22	1,00			
16	1,00	−0,22	−0,22	−0,99	0,12	1,00			
17	1,10	−0,12	−0,12	−1,00	0,02	1,00			
18	1,20	−0,02	−0,02	−1,00	−0,08	1,00			
19	1,30	0,08	0,08	−0,99	−0,18	1,00			
20	1,40	0,18	0,18	−0,97	−0,28	1,00			
21	1,50	0,28	0,28	−0,95	−0,37	1,00			
22	1,60	0,37	0,37	−0,91	−0,46	1,00			
23	1,70	0,46	0,46	−0,86	−0,55	1,00			
24	1,80	0,55	0,55	−0,81	−0,63	1,00			
25	1,90	0,63	0,63	−0,75	−0,70	1,00			
26	2,00	0,70	0,70	−0,68	−0,77	1,00			
27	2,10	0,77	0,77	−0,60	−0,83	1,00			
28	2,20	0,83	0,83	−0,52	−0,88	1,00			
29	2,30	0,88	0,88	−0,43	−0,93	1,00			
30	2,40	0,93	0,93	−0,33	−0,96	1,00			
31	2,50	0,96	0,96	−0,24	−0,98	1,00			
32	2,60	0,98	0,98	−0,14	−1,00	1,00			
33	2,70	1,00	1,00	−0,04	−1,00	1,00			
34	2,80	1,00	1,00	0,06	−1,00	1,00			
35	2,90	1,00	1,00	0,16	−0,98	1,00			
36	3,00	0,98	0,98	0,26	−0,95	1,00			
37	3,10	0,95	0,95	0,35	−0,92	1,00			
38	3,20	0,92	0,92	0,44	−0,87	1,00			
39	3,30	0,87	0,87	0,53	−0,82	1,00			
40	3,40	0,82	0,82	0,61	−0,76	1,00			
41	3,50	0,76	0,76	0,69	−0,69	1,00			
42	3,60	0,69	0,69	0,76	−0,61	1,00			
43	3,70	0,61	0,61	0,82	−0,53	1,00			
44	3,80	0,53	0,53	0,87	−0,45	1,00			
45	3,90	0,45	0,45	0,92	−0,35	1,00			
46	4,00	0,35	0,35	0,95	−0,26	1,00			
47	4,10	0,26	0,26	0,98	−0,16	1,00			
48	4,20	0,16	0,16	1,00	−0,06	1,00			
49	4,30	0,06	0,06	1,00	0,04	1,00			

▶ See example

Table of Contents

Table of Contents

Sreenshot: Experiment setup (credit:The Physics clasroom)

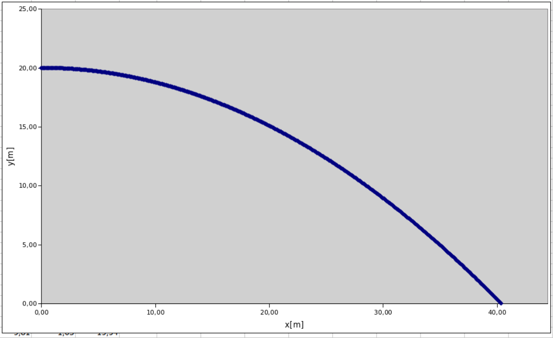


See example

Sreenshot: Spreadsheet approach

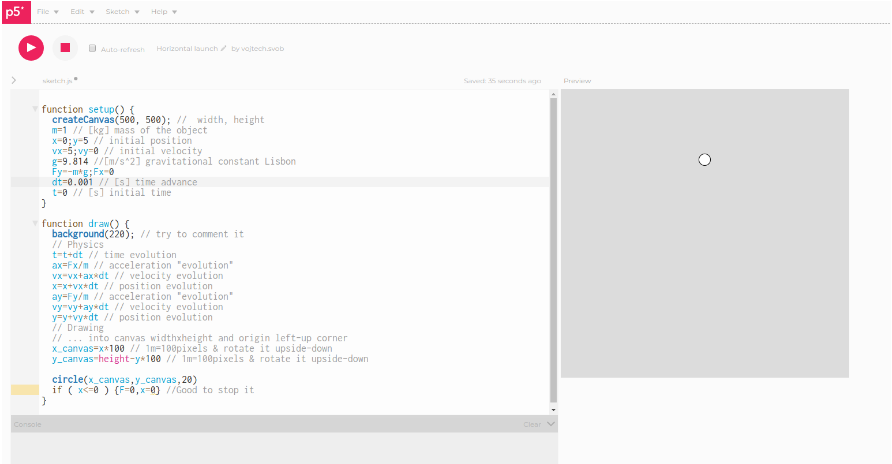
	A	B	C	D	E	F	G	H	I
1									
2	t	Fx	ax	vx	x	Fy	ay	vy	y
3	[s]	[N]	[m/s/s]	[m/s]	m	[N]	[m/s/s]	[m/s]	m
4	0,00	0,00	0,00	20,00	0,00	−9,81	−9,81	0,00	20,00
5	0,00	0,00	0,00	20,00	0,06	−9,81	−9,81	−0,03	20,00
6	0,01	0,00	0,00	20,00	0,12	−9,81	−9,81	−0,06	20,00
7	0,01	0,00	0,00	20,00	0,18	−9,81	−9,81	−0,09	20,00
8	0,01	0,00	0,00	20,00	0,24	−9,81			
9	0,01	0,00	0,00	20,00	0,30	−9,81			
10	0,02	0,00	0,00	20,00	0,36	−9,81			
11	0,02	0,00	0,00	20,00	0,42	−9,81			
12	0,02	0,00	0,00	20,00	0,48	−9,81			
13	0,03	0,00	0,00	20,00	0,54	−9,81			
14	0,03	0,00	0,00	20,00	0,60	−9,81			
15	0,03	0,00	0,00	20,00	0,66	−9,81			
16	0,04	0,00	0,00	20,00	0,72	−9,81			
17	0,04	0,00	0,00	20,00	0,78	−9,81			
18	0,04	0,00	0,00	20,00	0,84	−9,81			
19	0,05	0,00	0,00	20,00	0,90	−9,81			
20	0,05	0,00	0,00	20,00	0,96	−9,81			
21	0,05	0,00	0,00	20,00	1,02	−9,81			
22	0,05	0,00	0,00	20,00	1,08	−9,81			
23	0,06	0,00	0,00	20,00	1,14	−9,81			
24	0,06	0,00	0,00	20,00	1,20	−9,81			
25	0,06	0,00	0,00	20,00	1,26	−9,81			
26	0,07	0,00	0,00	20,00	1,32	−9,81			
27	0,07	0,00	0,00	20,00	1,38	−9,81			
28	0,07	0,00	0,00	20,00	1,44	−9,81			
29	0,08	0,00	0,00	20,00	1,50	−9,81			
30	0,08	0,00	0,00	20,00	1,56	−9,81			
31	0,08	0,00	0,00	20,00	1,62	−9,81			
32	0,08	0,00	0,00	20,00	1,68	−9,81			
33	0,09	0,00	0,00	20,00	1,74	−9,81			
34	0,09	0,00	0,00	20,00	1,80	−9,81			
35	0,09	0,00	0,00	20,00	1,86	−9,81			
36	0,10	0,00	0,00	20,00	1,92	−9,81			
37	0,10	0,00	0,00	20,00	1,98	−9,81			
38	0,10	0,00	0,00	20,00	2,04	−9,81			
39	0,11	0,00	0,00	20,00	2,10	−9,81			

	J	K	L	M	N	O	P	Q	R	S	T
1		Parameters									
2		m	1,000	kg	Fx	0			T_fall	2,02	
3		g	9,810	m/s/s		Fy	−9,81		x_fall	40,39	
4		dt	0,003	s						44,42	



▸ See example

Sreenshot: Processing approach



▸ See example

Table of Contents

Runge Kutta method

Let an initial value problem be specified:

$$\dot{y} = f(t, y), \quad y(t_0) = y_0$$

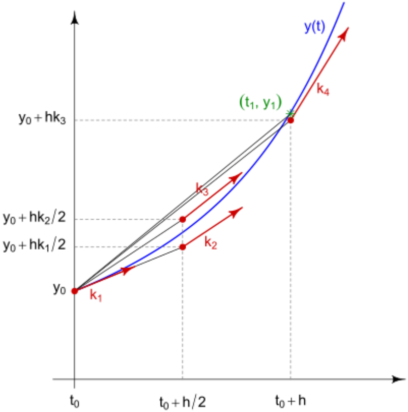


Figure: Slopes used by the classical Runge-Kutta method [Wik20e]

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$
$$t_{n+1} = t_n + h$$

$$k_1 = h\ f(t_n, y_n),$$
$$k_2 = h\ f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$
$$k_3 = h\ f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$
$$k_4 = h\ f\left(t_n + h, y_n + k_3\right).$$

Runge-Kutta versus Euler method

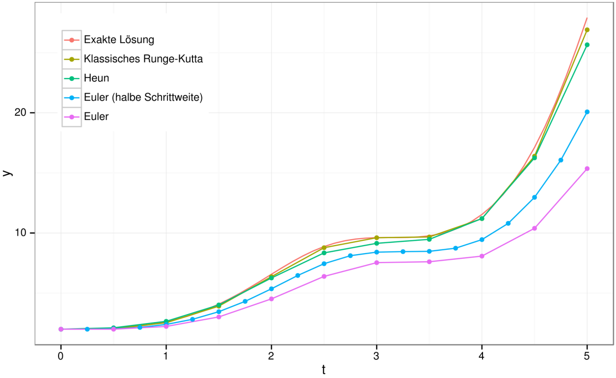


Figure: Runge-Kutta methods for the differential equation $y' = sin(t)^2 \cdot y$ [Wik20e]

Table of Contents

Sreenshot: odeint: Python solver

jupyter model (autosaved)

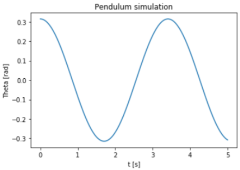
File Edit View Insert Cell Kernel Widgets Help

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
```

In [10]:
```
g=9.81
l=2.85
b=0.0 #With friction
def dTheta_dt(Theta, t):
    return [Theta[1], -b*Theta[1] -g/l*np.sin(Theta[0])]
Theta0 = [np.pi/10, 0]
t = np.linspace(0, 5, 200)
ThetaSolution = odeint(dTheta_dt, [np.pi/10, 0], t)
ThetaDraw = ThetaSolution[:,0]
T=2*np.pi*np.sqrt(l/g);print("T=%2.2f s"%T)
```

```
T=3.39 s
```

In [11]:
```
plt.xlabel("t [s]")
plt.ylabel("Theta [rad]")
plt.title("Pendulum simulation")
plt.plot(t,ThetaDraw);
```



See example

Table of Contents

Foucalt pendulum

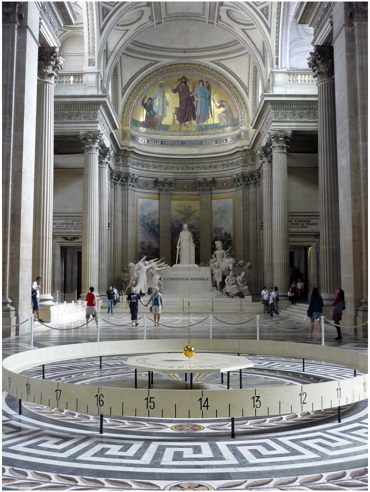


Figure: [Wik20b]

Foucault pendulum - dynamic equations

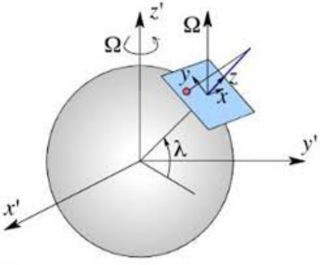


Figure: Foucault pendulum - setup

Coriolis force:

$$F_{c,x} = 2m\Omega \frac{dy}{dt} \sin\varphi$$
$$F_{c,y} = -2m\Omega \frac{dx}{dt} \sin\varphi$$

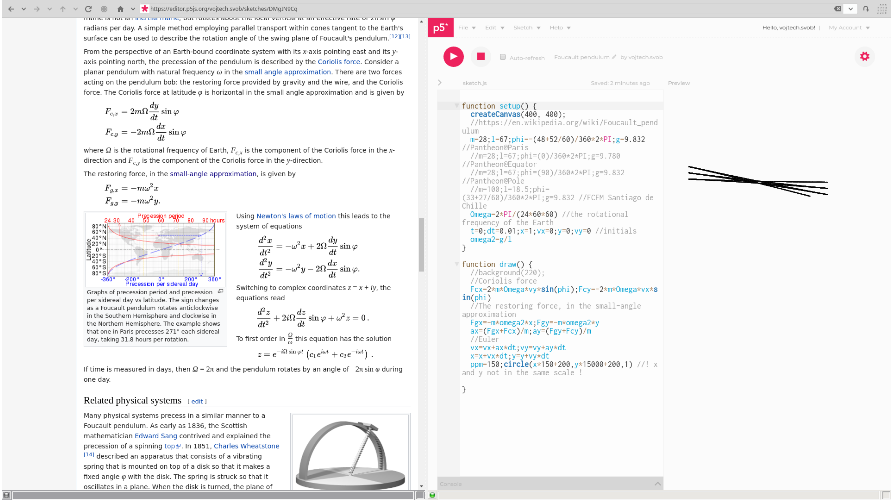
Restoring force (small angle approximation):

$$F_{g,x} = -m\omega^2 x$$
$$F_{g,y} = -m\omega^2 y.$$

Then dynamic equations:

$$\frac{d^2x}{dt^2} = -\omega^2 x + 2\Omega \frac{dy}{dt} \sin\varphi$$
$$\frac{d^2y}{dt^2} = -\omega^2 y - 2\Omega \frac{dx}{dt} \sin\varphi.$$

Sreenshot: Foucault pendulum @ processing



https://editor.p5js.org/vojtech.svob/sketches/DMgIN9Cq

frame is not an inertial frame, but rotates about the local vertical at an effective rate of $2\pi \sin \varphi$ radians per day. A simple method employing parallel transport within cones tangent to the Earth's surface can be used to describe the rotation angle of the swing plane of Foucault's pendulum.[12][13]

From the perspective of an Earth-bound coordinate system with its x-axis pointing east and its y-axis pointing north, the precession of the pendulum is described by the Coriolis force. Consider a planar pendulum with natural frequency ω in the small angle approximation. There are two forces acting on the pendulum bob: the restoring force provided by gravity and the wire, and the Coriolis force. The Coriolis force at latitude φ is horizontal in the small angle approximation and is given by

$$F_{c,x} = 2m\Omega \frac{dy}{dt} \sin\varphi$$
$$F_{c,y} = -2m\Omega \frac{dx}{dt} \sin\varphi$$

where Ω is the rotational frequency of Earth, $F_{c,x}$ is the component of the Coriolis force in the x-direction and $F_{c,y}$ is the component of the Coriolis force in the y-direction.

The restoring force, in the small-angle approximation, is given by

$$F_{g,x} = -m\omega^2 x$$
$$F_{g,y} = -m\omega^2 y.$$

Graphs of precession period and precession per sidereal day vs latitude. The sign changes as a Foucault pendulum rotates anticlockwise in the Southern Hemisphere and clockwise in the Northern Hemisphere. The example shows that one in Paris precesses 271° each sidereal day, taking 31.8 hours per rotation.

Using Newton's laws of motion this leads to the system of equations

$$\frac{d^2x}{dt^2} = -\omega^2 x + 2\Omega \frac{dy}{dt}\sin\varphi$$
$$\frac{d^2y}{dt^2} = -\omega^2 y - 2\Omega \frac{dx}{dt}\sin\varphi.$$

Switching to complex coordinates $z = x + iy$, the equations read

$$\frac{d^2z}{dt^2} + 2i\Omega \frac{dz}{dt}\sin\varphi + \omega^2 z = 0\,.$$

To first order in $\frac{\Omega}{\omega}$ this equation has the solution

$$z = e^{-i\Omega \sin\varphi t}\left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right).$$

If time is measured in days, then $\Omega = 2\pi$ and the pendulum rotates by an angle of $-2\pi \sin\varphi$ during one day.

Related physical systems [edit]

Many physical systems precess in a similar manner to a Foucault pendulum. As early as 1836, the Scottish mathematician Edward Sang contrived and explained the precession of a spinning top. In 1851, Charles Wheatstone [14] described an apparatus that consists of a vibrating spring that is mounted on top of a disk so that it makes a fixed angle φ with the disk. The spring is struck so that it oscillates in a plane. When the disk is turned, the plane of

p5* File Edit Sketch Help Hello, vojtech.svob! | My Account

Auto-refresh Foucault pendulum by vojtech.svob

sketch.js Saved: 2 minutes ago Preview

```
function setup() {
  createCanvas(400, 400);
  //https://en.wikipedia.org/wiki/Foucault_pendulum
  m=28;l=67;phi=-(48+52/60)/360*2*PI;g=9.832 //Pantheon@Paris
  //m=28;l=67;phi=(0)/360*2*PI;g=9.780 //Pantheon@Equator
  //m=28;l=67;phi=(90)/360*2*PI;g=9.832 //Pantheon@Pole
  //m=100;l=18.5;phi=(33+27/60)/360*2*PI;g=9.832 //FCFM Santiago de Chille
  Omega=2*PI/(24*60*60) //the rotational frequency of the Earth
  t=0;dt=0.01;x=1;vx=0;y=0;vy=0 //initials
  omega2=g/l
}

function draw() {
  //background(220);
  //Coriolis force
  Fcx=2*m*Omega*vy*sin(phi);Fcy=-2*m*Omega*vx*sin(phi)
  //The restoring force, in the small-angle approximation
  Fgx=-m*omega2*x;Fgy=-m*omega2*y
  ax=(Fgx+Fcx)/m;ay=(Fgy+Fcy)/m
  //Euler
  vx=vx+ax*dt;vy=vy+ay*dt
  x=x+vx*dt;y=y+vy*dt
  ppm=150;circle(x*150+200,y*15000+200,1) //! x and y not in the same scale !

}
```

Console

See example

Table of Contents

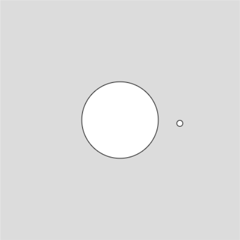
Sreenshot: Satellite motion @ processing

p5* File Edit Sketch Help

Auto-refresh Satellite motion by vojtech.svob

sketch.js Saved: 25 seconds ago Preview

```
function setup() {
  createCanvas(400, 400);
  kappa=6.672E-11
  m=1
  M=5.972E24
  //Initial conditions
  t=0;dt=100;x=10E6;y=0;vx=0;vy=7.5E3
}

function draw() {
  background(220);
  r=sqrt(sq(x)+sq(y))
  Fg=kappa*m*M/(sq(r))
  Fx=-Fg*(x/r);Fy=-Fg*(y/r)
  ax=Fx/m;ay=Fy/m
  vx=vx+ax*dt;vy=vy+ay*dt
  x=x+vx*dt;y=y+vy*dt
  t=t+dt

  mpp=100000
  circle(200,200,2*6.378E6/mpp)
  if (r>6.378E6) {circle(200+x/mpp,200+y/mpp,10)}
  //circle(200+x/mpp,200+y/mpp,10)

}
```



See example

Table of Contents

To be continued..

Thank you

for your attention

Tom Henderson.
The physics classroom: Motion of a mass on a spring.
https://www.physicsclassroom.com/class/waves/Lesson-0/Motion-of-a-Mass-on-a-Spring,
2020.

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